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Generalized Networks: Networks Embedded on a Matroid, Part II

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This paper represents the second part of the paper with the same title which appeared in *Networks*, Volume 6, Number 1. In this part we apply the theory presented in Part I to define a generalized network and derive its important properties.

3. RESISTANCE NETWORKS AND GENERALIZED NETWORKS

In this section we introduce the concept of a generalized resistance network and study some of its properties. The generalized network is an extension of the concepts of ordinary p-port resistance and RLC networks to matroids. As discussed in Part I, though we carry out the analysis in terms of resistance networks, the application to RLC networks is valid and given immediately by simple substitution.

3.1 Resistance Networks

In this subsection we describe a p-port resistance network as a preliminary to the discussion of the generalized resistance network in the next subsection. A p-port resistance network is an interconnection of two types of elements: port elements and resistance elements. A port element is denoted by a directed edge (Figure 11) and the convention used is that the direction of positive current (i) coincides with the direction of the arrow. Positive potential difference (v) means that the arrow points from the vertex of high potential to the vertex of low potential. Note that the product vi represents the instantaneous power delivered *to* the port element. A port edge may thus be considered as a *distinguished* edge. A resistance element is identical to a port element with the additional requirement that $v = iz$, where $0 < z < \infty$ and z is called the *resistance* of the element.

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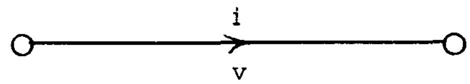

Fig. 11 Network Element.

The resistance and port elements are interconnected in some manner to form a network and this interconnection is represented by an oriented graph which is called the *network graph.*

If the network consists of n elements, p of which are port elements and n-p resistance elements, then let

$$\underline{v}_b^t = [v_1 \ldots v_{n-p}]$$

and

$$\underline{i}_b^t = [i_1 \ldots i_{n-p}]$$

be the vectors of resistance voltages and currents, respectively. Similarly

$$\underline{v}_p^t = [v_{n-p+1} \ldots v_n]$$

and

$$\underline{i}_p^t = [i_{n-p+1} \ldots i_n]$$

are the vectors of port voltages and currents, respectively. Also with each resistance element there is an associated resistance z_i, for $i = 1, \ldots, n-p$. The matrix

$$Z_b = \text{diag}[z_1, \ldots, z_{n-p}]$$

is called the *resistance-element impedance matrix.* The vectors $\underline{v}_b$ and $\underline{i}_b$ must satisfy

$$\underline{v}_b = Z_b \underline{i}_b \tag{1a}$$

or

$$\underline{i}_b = Y_b \underline{v}_b \ , \tag{1b}$$

where $Y_b = Z_b^{-1}$. Y_b is called the *resistance-element admittance matrix.* Let

$$\underline{v} = \begin{bmatrix} \underline{v}_b \\ --- \\ \underline{v}_p \end{bmatrix} \quad \text{and} \quad \underline{i} = \begin{bmatrix} \underline{i}_b \\ --- \\ \underline{i}_p \end{bmatrix} .$$

The vectors $\underline{v}$ and $\underline{i}$ are called the *network voltage* and *current* vectors, respectively.

The vectors $\underline{i}$ and $\underline{v}$ are not only required to satisfy (1a) or (1b) but in addition must satisfy *Kirchhoff's current law* (KCL) and *Kirchhoff's voltage law* (KVL). Thus the algebraic sum of the currents in any bond of the network graph must be zero and the algebraic sum of the voltages around any polygon in the network must be zero. These last two constraints on the network voltage and current vectors are called *topological constraints* and the relation (1a) is called an *Ohm's law* constraint.

The topological constraints on the network voltage and current vectors can be stated differently. Let G be the network graph of a p-port resistance network and I and V the 1-cycle space and the coboundary space, respectively, of G over the field F. Then $\underline{i}$ satisfies KCL if and only if $\underline{i}^t$ is the representative vector of some member of I and $\underline{v}$ satisfies KVL if and only if $\underline{v}^t$ is the representative vector of some member of V. Consequently Kirchhoff's laws can be written symbolically as

$$\underline{i} \in I \quad \text{(KCL)} \tag{2}$$

and

$$\underline{v} \in V \quad \text{(KVL)}. \tag{3}$$

The equations (1a) or (1b), (2) and (3) are called the *network equations*.

In order to retain the familiar properties of Z, the *open-circuit* (o.c.) *impedance matrix*, and Y, the *short-circuit* (s.c.) *admittance matrix*, of a p-port resistance network we define an auxiliary port-voltage vector $\underline{e}_p = -\underline{v}_p$. Then the o.c. impedance matrix of a network exists if for *any* prescribed set of port currents $\underline{i}_p$ the network equations uniquely determine the response $\underline{e}_p$. Similarly the s.c. admittance matrix of a network exists if for *any* prescribed set of port voltages $\underline{e}_p$ the network equations uniquely determine the response $\underline{i}_p$. If Z exists, then the network operation, viewed from the ports, can be expressed as

$$\underline{e}_p = Z\underline{i}_p$$

and if Y exists, then

$$\underline{i}_p = Y\underline{e}_p .$$

Previously in Section 1 we indicated that certain matrices were related to p-port resistance networks. At this time, we give the precise definitions for two of these matrices.

A symmetric matrix of real numbers whose main-diagonal elements are greater than or equal to the sum of the absolute magnitudes of all the other elements in the same row (column) is called a *dominant* matrix.

A $p \times p$ symmetric matrix of real numbers is called a *paramount* matrix if every principal minor of order r is greater than or equal to the absolute value of any rth-order minor formed from the same rows (columns) for $r = 1, \ldots, p-1$.

The rest of this section is devoted to the generalized resistance network and its bearing on p-port resistance networks.

3.2 *Generalized Networks*

In this subsection we define a resistance network on a regular matroid. As in the case of p-port resistance networks we will consider the generalized network to be an interconnection of two kinds of elements: resistance elements and port elements. In general the generalized network will consist of n elements, p of which are port elements and n-p resistance elements.

Let $\mathcal{M} = (\mathcal{C}, E)$ be a *regular* matroid on a finite set E. The set E is partitioned into two sets E_p and E_b. The elements in E_p are the port elements and the elements in E_b the resistance elements. Enumerate the elements of E such that

$$E = E_b \cup E_p ,$$

where

$$E_b = \{e_1, e_2, \ldots, e_{n-p}\}$$

and

$$E_p = \{e_{n-p+1}, \ldots, e_n\} .$$

With each element e_i in E we associate two variables u_i and w_i (for $i = 1, \ldots, n$). We define the vectors $\underline{u}$ and $\underline{w}$ as follows:

$$\underline{u} = \begin{bmatrix} \underline{u}_b \\ \hline \underline{u}_p \end{bmatrix} \quad \text{and} \quad \underline{w} = \begin{bmatrix} \underline{w}_b \\ \hline \underline{w}_p \end{bmatrix} ,$$

where

$$\underline{w}_b^t = [w_1 \ldots w_{n-p}]$$

$$\underline{w}_p^t = [w_{n-p+1} \ldots w_n],$$

$$\underline{u}_b^t = [u_1 \ldots u_{n-p}]$$

and

$$\underline{u}_p^t = [u_{n-p+1} \ldots u_n].$$

We have chosen not to use $\underline{v}$ and $\underline{i}$ as variables in the generalized case in order to give a single analysis of the generalized network which later can be specialized to an impedance and/or admittance formulation.

As in Subsection 3.1, we associate with each member of E_b a positive number d_i $(i=1,\ldots,n-p)$ and require that

$$\underline{w}_b = D\,\underline{u}_b,$$

where

$$D = \mathrm{diag}[d_1, d_2, \ldots, d_{n-p}].$$

D is called the *resistance-element immittance matrix.*

The next step in defining a generalized network is to write the "topological" constraints for the vectors $\underline{u}$ and $\underline{w}$. Since M is regular, there exists a regular vector space R on E over the field of real numbers such that the supports of the primitive vectors of R are in 1-1 correspondence with the circuits of M, that is, $M = M_R$. The vector space R is not unique but since R is a regular vector space, we think of choosing a particular R as fixing the relative orientation of the elements in each of the circuits of M. To see this choose $C \in C_R$; then there exists a primitive vector $f \in R$ such that

$$\|f\| = C.$$

The nonzero values of f are either ±1 and therefore can be used to determine the relative orientation of the members of C. Thus if $f(e_i) = f(e_j) = \pm 1$, we say that e_i and e_j are *similarly oriented in C* and if $f(e_k) = -f(e_s) = \pm 1$, we say that e_k and e_s are *oppositely oriented in C*. By (2.2-4), the choice of R *uniquely* determines the relative orientations of the elements in C.

Generalizing KCL and KVL, we require that $\underline{u}^t$ be the representative vector of some member of $\mathcal{R}$ and $\underline{w}^t$ be the representative vector of some member of $\perp\mathcal{R}$, the complementary orthogonal subspace of $\mathcal{R}$. We write the generalized KCL and KVL symbolically as

$$\underline{u} \in \mathcal{R}$$

and

$$\underline{w} \in \perp\mathcal{R}.$$

We define a *generalized network* N as follows:

$$N = (M_{\mathcal{R}}, \mathcal{R}, D; E),$$

where $M_{\mathcal{R}}$ is a *regular* matroid on a finite set E and $\mathcal{R}$ is a corresponding *regular* vector space on E over the field of real numbers.

The *generalized network equations* are

$$\underline{u} \in \mathcal{R}, \qquad (1)$$

$$\underline{w} \in \perp\mathcal{R}, \qquad (2)$$

$$\underline{w}_b = D\,\underline{u}_b, \qquad (3)$$

where $D = \text{diag}[d_1, \ldots, d_{n-p}]$.

Equations (1) and (2) are the "topological" constraints on $\underline{u}$ and $\underline{w}$ while Equation (3) is an Ohm's law constraint.

At this point we will make the appropriate correspondences between the generalized network and the ordinary impedance and admittance formulations of p-port resistance networks.

Let G be a network graph of a p-port resistance network and partition E(G) according to port and resistance designations. Thus

$$E(G) = E(G)_b \cup E(G)_p,$$

where

$$E(G)_b = \{e_1, \ldots, e_{n-p}\}$$

and

$$E(G)_p = \{e_{n-p+1}, \ldots, e_n\}.$$

The edges in $E(G)_b$ correspond to the resistances and the edges in $E(G)_p$ correspond to the ports. The quantities $\underline{i}^t = (\underline{i}_b^t, \underline{i}_p^t)$, $\underline{v}^t = (\underline{v}_b^t, \underline{v}_p^t)$ and $Z_b = \text{diag}[z_1, \ldots, z_{n-p}]$, where $0 < z_i < \infty$ for $i = 1, \ldots, n-p$, are defined in Subsection 3.1. There are two possible ways to make a correspondence between generalized networks and p-port resistance networks.

Consider the following correspondence. Suppose

$$\underline{u} = \underline{i}. \tag{4}$$

Then it follows that

$$\underline{w} = \underline{v}, \tag{5}$$

$$\mathcal{R} = I, \tag{6}$$

$$M_{\mathcal{R}} = \mathcal{P}(G) \tag{7}$$

and

$$D = Z_b, \tag{8}$$

where I is the 1-cycle space of G over the field of real numbers. Thus the requirement that $\underline{u}$ corresponds to $\underline{i}$ determines the generalized network

$$N_Z = (\mathcal{P}(G),\ I,\ Z_b;\ E(G)).$$

If one chooses to have $\underline{v}$ correspond to $\underline{u}$, the generalized network N_Y is obtained:

$$N_Y = (\mathcal{B}(G),\ V,\ Y_b;\ E(G)).$$

V is, of course, the coboundary space of G over the field of real numbers.

The subscripts Z and Y reflect the fact that N_Z will lead to an impedance formulation and N_Y yields an admittance formulation. The correspondences between generalized networks and p-port resistance networks are listed, for future reference, in Table 3-1.

Boesch [21] has shown convenient formulas for Z and Y in terms of topological matrices associated with the network graph and in the next subsection the generalized network is subjected to a similar analysis and the results are interpreted in terms of matroid structure.

Generalized Networks	Network Equations
$N = (M_{\mathcal{R}}, \mathcal{R}, D; E)$	(i) $\underline{u} \varepsilon \mathcal{R}$ (ii) $\underline{w} \varepsilon \perp\mathcal{R}$ (iii) $\underline{w}_b = D\underline{u}_b$
$N_Z = (\mathcal{P}(G), I, Z_b; E(G))$	(i) $\underline{i} \varepsilon I$ (ii) $\underline{v} \varepsilon V$ (iii) $\underline{v}_b = Z_b\underline{i}_b$
$N_Y = (\mathcal{B}(G), V, Y_b; E(G))$	(i) $\underline{v} \varepsilon V$ (ii) $\underline{i} \varepsilon I$ (iii) $\underline{i}_b = Y_b\underline{v}_b$

Table 3-1 Table of Correspondences.

3.3 *Analysis of Generalized Networks*

Having defined a generalized network, the next question to answer is: "how does it work?" Can we take it apart to show "what makes it tick?" In other words, if we specify $\underline{u}_p$, how do the network equations determine $\underline{u}$ and $\underline{w}$. We first introduce some definitions and notation.

A network $N = (M_{\mathcal{R}}, \mathcal{R}, D; E)$ is called *nondegenerate* if one can specify $\underline{u}_p$ arbitrarily and this specification, along with the network equations, uniquely determines $\underline{u}$ and $\underline{w}$. Let $\mathcal{N}$ denote the class of nondegenerate networks.

Suppose $f \varepsilon \mathcal{R}$ and $\underline{x}^t$ is a representative vector for f. We define

$$\|\underline{x}\| = \|f\|.$$

Also, as was done in the network equations, we write

$$\underline{x} \varepsilon \mathcal{R}$$

to mean that there exists a vector f ε R such that $\underline{x}^t$ is the representative vector of f. We call $\underline{x}$ elementary (primitive) if there exists an elementary (primitive) vector f in R such that $\underline{x}^t$ is the representative vector for f.

The next theorem is the main theorem of this subsection and it characterizes, in terms of matroid structure, those generalized networks which are nondegenerate. Moreover, in the course of proving (3.3-1) we derive explicit expressions for the "response" of a nondegenerate generalized network to an arbitrary port vector $\underline{u}_p$.

Theorem (3.3-1): A network $N = (M_R, R, D; E)$ is in N if and only if E_p contains no circuit of M_R^*.

Proof: Let N ε N. Then $\underline{u}_p$ can be specified arbitrarily. Assume that there is a circuit C of M_R^* such that $C \subseteq E_p$. Then there exists a primitive vector $\underline{x}$ $\varepsilon \perp R$ such that

$$\|\underline{x}\| = C \subseteq E_p.$$

Since $\underline{u}$ ε R, it follows that

$$\underline{x}^t \; \underline{u} = 0.$$

Moreover, since $\|\underline{x}\| \subseteq E_p$, there exists a linear relation among the coordinates of $\underline{u}_p$. This contradicts the hypothesis that N ε N and accordingly no circuit of M_R^* is contained in E_p.

To show sufficiency, suppose that no circuit of M_R^* is contained in E_p. Let R^* be a representative matrix for $\perp R$. Since $\underline{u}$ ε R, it follows that

$$R^* \; \underline{u} = \underline{0}. \tag{1}$$

Also since $\underline{w}$ $\varepsilon \perp R$, $\underline{w}$ can be expressed as some linear combination of the rows of R^*, that is,

$$\underline{w} = R^{*t} \; \underline{\varphi}, \tag{2}$$

where

$$\underline{\varphi} = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_r \end{bmatrix}$$

and

$$r = \text{dimension } (\perp\mathcal{R}).$$

If we partition R^* according to $\underline{u}_b$ and $\underline{u}_p$, it follows from (1) that

$$R_b^* \underline{u}_b + R_p^* \underline{u}_p = \underline{0}, \tag{3}$$

where

$$R^* = [R_b^* \vdots R_p^*].$$

Using $\underline{u}_b = D^{-1} \underline{w}_b$ (D^{-1} exists since $d_i > 0$ for $i = 1, \ldots, n-p$) in (3) we get

$$R_b^* D^{-1} \underline{w}_b = - R_p^* \underline{u}_p. \tag{4}$$

From (2) it follows that

$$\underline{w}_b = R_b^{*t} \underline{\varphi} \tag{5}$$

and

$$\underline{w}_p = R_p^{*t} \underline{\varphi}. \tag{6}$$

Substituting (5) into (4), we get

$$[R_b^* D^{-1} R_b^{*t}] \underline{\varphi} = - R_p^* \underline{u}_p. \tag{7}$$

We show by contradiction that the hypothesis implies $\det[R_b^* D^{-1} R_b^{*t}] \neq 0$.

Assume $\det[R_b^* D^{-1} R_b^{*t}] = 0$. Applying the Binet-Cauchy formula [35] twice to $\det[R_b^* D^{-1} R_b^{*t}]$, we get

$$\det[R_b^* D^{-1} R_b^{*t}] = \sum D^{-1} \begin{pmatrix} i_1,\ldots,i_r \\ i_1,\ldots,i_r \end{pmatrix} \left[R_b^* \begin{pmatrix} 1,\ldots,r \\ i_1,\ldots,i_r \end{pmatrix}\right]^2 . \quad (8)$$

$$1 \leq i_1 < \ldots < i_r \leq n-p$$

If B is a matrix, the notation

$$B \begin{pmatrix} i_1,\ldots,i_r \\ j_1,\ldots,j_r \end{pmatrix}$$

represents the determinant of the submatrix of B formed by using rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$.

Since $\det[R_b^* D^{-1} R_b^{*t}] = 0$, it is clear from (8) that the rank of R_b^* is less than r. Consequently if $S \subseteq E_b$ and $\alpha(S) = r$, $\det[R_b^*(S)] = 0$. Therefore (see Table 2-2, line 2) no $S \subseteq E_b$ is a base of $M_{\mathcal{R}}$. By (2.4-2), E_p is not contained in any base of $M_{\mathcal{R}}^*$. Therefore E_p contains a circuit of $M_{\mathcal{R}}^*$; but this contradicts the hypothesis. Accordingly $\det[R_b^* D^{-1} R_b^{*t}] \neq 0$ and from (7) we get

$$\underline{\varphi} = -[R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \underline{u}_p . \quad (9)$$

Substituting (9) into (5) and (6) and using $\underline{u}_b = D^{-1} \underline{w}_b$, we get the following results:

$$\underline{w} = \begin{bmatrix} \underline{w}_b \\ \hline \underline{w}_p \end{bmatrix} = \begin{bmatrix} -R_b^{*t}[R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \\ \hline -R_p^{*t}[R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \end{bmatrix} \underline{u}_p \quad (10)$$

$$\underline{u} = \begin{bmatrix} \underline{u}_b \\ \hline \underline{u}_p \end{bmatrix} = \begin{bmatrix} -D^{-1} R_b^{*t} [R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \\ \hline 1_p \end{bmatrix} \underline{u}_p . \quad (11)$$

Equations (10) and (11) show the explicit dependence of $\underline{u}$ and $\underline{w}$ on $\underline{u}_p$. Since (10) and (11) were obtained by applying necessary conditions and are invariant with respect to the choice of R^*, it follows that specification of $\underline{u}_p$ uniquely determines $\underline{u}$ and $\underline{w}$. Therefore N ε $\mathcal{N}$. □

An immediate corollary of (3.3-1) is

Theorem (3.3-2): Let $N = (M_R, \mathcal{R}, D; E) \in \mathcal{N}$ *and* R^* *be a representative matrix for* $\perp\mathcal{R}$. *Partition* R^* *as* $R^* = [R_b^* | R_p^*]$, *where* R_b^* *and* R_p^* *correspond to the resistance and port elements, respectively. Then*

$$\underline{w} = \begin{bmatrix} \underline{w}_b \\ \hline \underline{w}_p \end{bmatrix} = \begin{bmatrix} -R_b^{*t}[R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \\ \hline -R_p^{*t}[R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \end{bmatrix} \underline{u}_p$$

and

$$\underline{u} = \begin{bmatrix} \underline{u}_b \\ \hline \underline{u}_p \end{bmatrix} = \begin{bmatrix} -D^{-1} R_b^{*t} [R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \\ \hline 1_p \end{bmatrix} \underline{u}_p .$$

The *immittance matrix* X_N is defined as

$$X_N = R_p^{*t}[R_b^* D^{-1} R_b^{*t}]^{-1} R_p^*.$$

Therefore X_N characterizes the "operation" of the generalized network in terms of a port description, that is,

$$\underline{w}_p = - X_N \underline{u}_p .$$

Figure 12 depicts the port description of a generalized network.

An alternate characterization of a nondegenerate network is given by the following theorem which is a consequence of Theorem (3.3-1).

Theorem (3.3-3): Let $N = (M_R, \mathcal{R}, D; E)$ *and* R^* *be a representative matrix for* $\perp\mathcal{R}$. *Partition* R^* *as* $R^* = [R_b^* | R_p^*]$, *where* R_b^* *and and* R_p^* *correspond to the resistance and port elements, respectively. Then* N *is in* $\mathcal{N}$ *if and only if rank* $(R_b^*) =$ *rank* (R^*).

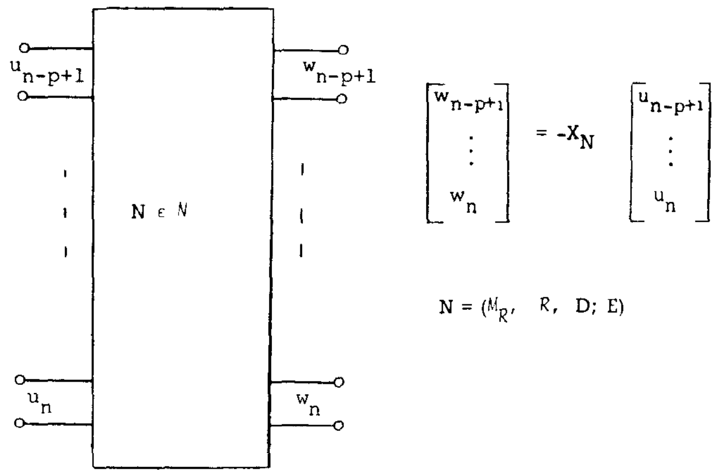

Fig. 12 Generalized Network.

The beauty of matroid theory becomes apparent as one realizes that the matroid structure allows one to visualize the "interconnection" of the elements in E of a generalized network $N = (\mathcal{M}_{\mathcal{R}}, \mathcal{R}, D; E)$. Theorem (3.3-1) is an excellent example of this since it gives the existence of X_N in terms of the matroid structure. Also matroid theory eliminates the necessity of thinking in terms of admittance or impedance and thus focuses attention on the essential features of the analysis of p-port resistance networks. As we will discuss later, however, the different matroid classes, as depicted in Figure 10, allow us to distinguish in a precise way the *differences* between the admittance and impedance formulations of p-port resistance networks.

Let us return now to Table 3-1 and interpret X_{N_Z} and X_{N_Y}. It is easy to see that $X_{N_Z} = Z$, the o.c. impedance matrix for the resistance network, and $X_{N_Y} = Y$, the s.c. admittance matrix.

$N = (\mathcal{M}_{\mathcal{R}}, \mathcal{R}, D; E)$	$N_Z = (\mathcal{P}(G), I, Z_b; E(G))$	$N_Y = (\mathcal{B}(G), V, Y_b; E(G))$
$\underline{w}_p = -X_N \underline{u}_p$	$-\underline{v}_p = X_{N_Z} \underline{i}_p$ $X_{N_Z} = Z$	$\underline{i}_p = -X_{N_Y} \underline{v}_p$ $X_{N_Y} = Y$

Table 3-2 Table of Correspondences.

To conclude this subsection we show how one obtains the known results on the existence of Z and Y as special cases of (3.3-1), namely, Z(Y) exists if and only if there exists no bond containing only current sources (there exists no polygon containing only voltage sources).

Theorem (3.3-4): Let G be the network graph of a p-port resistance network. Then Z(Y), the o.c. impedance (s.c. admittance) matrix, exists if and only if $G \times E(G)_p$ *(*$G \cdot E(G)_p$*) contains no bonds (polygons).*

Proof: By (3.3-1) and (2.4-3) X_{N_Z} exists if and only if $\mathcal{B}(G)$ has no circuits contained in $E(G)_p$. By the definition of a contraction, $\mathcal{B}(G)$ has no circuits in $E(G)_p$ if and only if $\mathcal{B}(G) \times E(G)_p$ has no circuits. By (2.1-9), $\mathcal{B}(G) \times E(G)_p = \mathcal{B}(G \times E(G)_p)$. Therefore $\mathcal{B}(G) \times E(G)_p$ has no circuits, and hence Z exists, if and only if $G \times E(G)_p$ contains no bonds.

The proof for Y follows the same pattern as that for Z. □

Example 5: Let G be the graph in Figure 13 and

$$E(G) = E(G)_b \cup E(G)_p,$$

where

$$E(G)_b = \{e_1, \ldots, e_5\}$$

and

$$E(G)_p = \{e_6, e_7, e_8\}.$$

Z, the o.c. impedance matrix, exists since $G \times E(G)_p$ contains no bonds (see Figure 14). Y, the s.c. admittance matrix, does not exist since $G \cdot E(G)_p$ contains a polygon (in this case a loop). $G \cdot E(G)_p$ is shown in Figure 15.

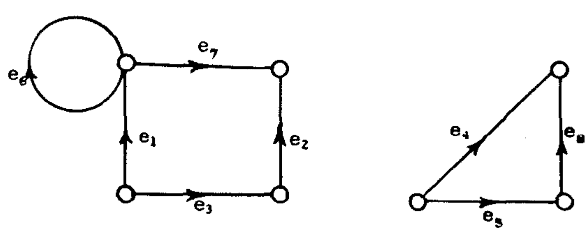

Fig. 13 Graph of Example 5.

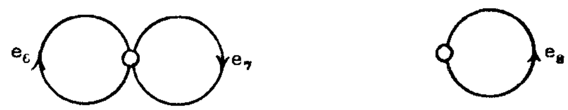

Fig. 14 $G \times E(G)_p$.

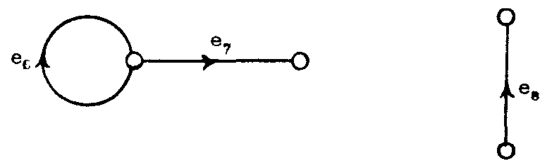

Fig. 15 $G \cdot E(G)_p$.

3.4 *Properties of X_N*

In this subsection we prove that if $N \in \mathcal{N}$, then X_N is a paramount matrix. The method of proof yields "topological" formulas for the generalized network and consequently extends the concept of a "topological" formula to matroids.

We also treat, in this subsection, the special case when $r(M_{\mathcal{R}}) = p$ and derive for this case an additional necessary condition on X_N.

To conclude this subsection we indicate that the modified topological matrices introduced by Cederbaum [31] for p-port resistance networks can be extended to generalized networks.

Theorem (3.4.1): If $N = (M_{\mathcal{R}}, \mathcal{R}, D; E)$ belongs to $\mathcal{N}$, then X_N is a paramount matrix.

Proof: $X_N = R_p^{*t}[R_b^* D^{-1} R_b^{*t}]^{-1} R_p^*$, where $R^* = [R_b^* \mid R_p^*]$ is a representative matrix for $\perp\mathcal{R}$, $R_b^* = r \times (n-p)$, $R_p^* = r \times p$ and $r = r(M_{\mathcal{R}}) = \text{dimension } (\perp\mathcal{R})$. By (3.3-3), $\det[R_b^* D^{-1} R_b^{*t}] \neq 0$ and consequently $n-p \geq r$.

Set

$$A = [R_b^* D^{-1} R_b^{*t}]^{-1}.$$

At this point we introduce a useful notation: let $\bar{i}_s$ represent the index set $i_1 < \cdots < i_s$. Then using the Binet-Cauchy formula [35] it is not difficult to show that

$$X_N\begin{pmatrix}\bar{i}_s\\ \bar{j}_s\end{pmatrix} = \sum_{\substack{1\le \bar{k}_s \le r\\ 1\le \bar{h}_s\le r}} A\begin{pmatrix}\bar{h}_s\\ \bar{k}_s\end{pmatrix} R_p^*\begin{pmatrix}\bar{h}_s\\ \bar{i}_s\end{pmatrix} R_p^*\begin{pmatrix}\bar{k}_s\\ \bar{j}_s\end{pmatrix} \tag{1}$$

for $s \le r$, and

$$X_N\begin{pmatrix}\bar{i}_s\\ \bar{j}_s\end{pmatrix} = 0 \tag{2}$$

for $s > r$. The summation in (1) is over all index sets $\bar{k}_s$ and $\bar{h}_s$ satisfying

$$1 \le k_1 < \cdots < k_s \le r$$

and

$$1 \le h_1 < \cdots < h_s \le r.$$

Furthermore [35]

$$A\begin{pmatrix}\bar{h}_s\\ \bar{k}_s\end{pmatrix} = \frac{(-1)^{\beta}[R_b^* D^{-1} R_b^{*t}]\begin{pmatrix}\overline{k'}_{r-s}\\ \overline{h'}_{r-s}\end{pmatrix}}{\det[R_b^* D^{-1} R_b^{*t}]} \tag{3}$$

for $s < r$, where $\beta = \sum_{i=1}^{s} (h_i+k_i)$. The indices $k_1' < \cdots < k_{r-s}'$ (i.e. $\overline{k'}_{r-s}$) and $k_1 < \cdots < k_s$ form a complete set of indices on $1, \ldots, r$. Similarly $\overline{h'}_{r-s}$ and $\bar{h}_s$ form a complete set of indices on $1, \ldots, r$.

When $s = r$

$$A\begin{pmatrix}\bar{h}_r\\ \bar{k}_r\end{pmatrix} = \frac{1}{\det[R_b^* D^{-1} R_b^{*t}]} . \tag{4}$$

Expand the right-hand side of (3) using the Binet-Cauchy formula. The result is

$$[R_b^* D^{-1} R_b^{*t}]\begin{pmatrix}\overline{k'}_{r-s}\\ \overline{h'}_{r-s}\end{pmatrix} = \sum_{1\le \bar{m}_{r-s}\le n-p} D^{-1}\begin{pmatrix}\bar{m}_{r-s}\\ \bar{m}_{r-s}\end{pmatrix} R_b^*\begin{pmatrix}\overline{k'}_{r-s}\\ \bar{m}_{r-s}\end{pmatrix} R_b^*\begin{pmatrix}\overline{h'}_{r-s}\\ \bar{m}_{r-s}\end{pmatrix} \tag{5}$$

for $s < r$. Substituting (5) and (3) into (1) we get:

$$X_N \begin{pmatrix} \bar{i}_s \\ \bar{j}_s \end{pmatrix} = \sum_{1 \leq \bar{m}_{r-s} \leq n-p} \frac{D^{-1} \begin{pmatrix} \bar{m}_{r-s} \\ \bar{m}_{r-s} \end{pmatrix}}{\det[R_b^* D^{-1} R_b^{*t}]} H K, \tag{6}$$

where

$$H = \sum_{1 \leq \bar{k}_s \leq r} (-1)^{\sum_{i=1}^{s} k_i} R_b^* \begin{pmatrix} \bar{k}'_{r-s} \\ \bar{m}_{r-s} \end{pmatrix} R_p^* \begin{pmatrix} \bar{k}_s \\ \bar{j}_s \end{pmatrix}$$

and

$$K = \sum_{1 \leq \bar{h}_s \leq r} (-1)^{\sum_{i=1}^{s} k_i} R_b^* \begin{pmatrix} \bar{h}'_{r-s} \\ \bar{m}_{r-s} \end{pmatrix} R_p^* \begin{pmatrix} \bar{h}_s \\ \bar{i}_s \end{pmatrix}$$

for $s < r$.

The terms H and K of (6) are easily seen to correspond to the Laplace expansion of minors formed from some r columns of R^*.

We introduce the useful notation:

$$R^*(\bar{m}_{r-s}|\bar{i}_s) = R^* \begin{pmatrix} 1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, r \\ m_1, \ldots, m_{r-s}, \; n-p+i_1, \ldots, n-p+i_s \end{pmatrix}.$$

Then (6) becomes

$$X_N \begin{pmatrix} \bar{i}_s \\ \bar{j}_s \end{pmatrix} = \sum_{1 \leq \bar{m}_{r-s} \leq n-p} \frac{D^{-1} \begin{pmatrix} \bar{m}_{r-s} \\ \bar{m}_{r-s} \end{pmatrix}}{\det[R_b^* D^{-1} R_b^{*t}]} R^*(\bar{m}_{r-s}|\bar{i}_s) \; R^*(\bar{m}_{r-s}|\bar{j}_s) \tag{7}$$

for $s < r$.

To obtain the case $s = r$, we combine (4) and (1) to get

$$X_N \begin{pmatrix} \bar{i}_r \\ \bar{j}_r \end{pmatrix} = \frac{R_p^* \begin{pmatrix} 1, \ldots, r \\ i_1, \ldots, i_r \end{pmatrix} R_p^* \begin{pmatrix} 1, \ldots, r \\ j_1, \ldots, j_r \end{pmatrix}}{\det[R_b^* D^{-1} R_b^{*t}]}. \tag{8}$$

Since $\perp R$ is a regular subspace (see 2.2-5) the minors of order r of R^* are restricted to the values $\pm L$ or 0, where L is some positive number. Also the terms $D^{-1}\begin{pmatrix}\bar{m}_{r-s}\\ \bar{m}_{r-s}\end{pmatrix}$ are positive. It should therefore be clear from (7) that a principal minor of order $s < r$ is greater than the absolute value of any s^{th}-order minor formed from the same rows. Moreover, consideration of Equations (2) and (4) permits one to conclude that X_N is in fact a paramount matrix.

We have the following corollary to (3.4-1)

Theorem (3.4-2): Let $N = (M_R, R, D; E) \in \mathcal{N}$ *and* R^* *be a representative matrix for* $\perp R$. *Partition* R^* *as* $R^* = [R^*_b | R^*_p]$, *where* R^*_b *and* R^*_p *correspond to the resistance and port elements, respectively. Then*

$$X_N\begin{pmatrix}\bar{i}_s\\ \bar{j}_s\end{pmatrix} = \sum_{1\le\bar{m}_{r-s}\le n-p} \frac{D^{-1}\begin{pmatrix}\bar{m}_{r-s}\\ \bar{m}_{r-s}\end{pmatrix}}{\Delta} R^*(\bar{m}_{r-s}|\bar{i}_s)\ R^*(\bar{m}_{r-s}|\bar{j}_s)$$

for $s < r$,

$$X_N\begin{pmatrix}\bar{i}_s\\ \bar{j}_s\end{pmatrix} = 0$$

for $s > r$ *and*

$$X_N\begin{pmatrix}\bar{i}_r\\ \bar{j}_r\end{pmatrix} = \frac{R^*_p\begin{pmatrix}1,\ldots,r\\ i_1,\ldots,i_r\end{pmatrix} R^*_p\begin{pmatrix}1,\ldots,r\\ j_1,\ldots,j_r\end{pmatrix}}{\Delta},$$

where $\Delta = \det[R^*_b D^{-1} R^{*t}_b]$.

It should be apparent that from (3.4-2) we can obtain "topological" formulas for the generalized network. Let us consider the case $s < r$ and $(X_N)_{i,j}$, the i,j element of X_N:

$$(X_N)_{i,j} = \frac{\displaystyle\sum_{1\le \bar{m}_{r-1}\le n-p} D^{-1}\begin{pmatrix}\bar{m}_{r-1}\\ \bar{m}_{r-1}\end{pmatrix} \frac{R^*(\bar{m}_{r-1}|i)}{L} \frac{R^*(\bar{m}_{r-1}|j)}{L}}{\displaystyle\sum_{1\le \bar{h}_r\le n-p} D^{-1}\begin{pmatrix}\bar{h}_r\\ \bar{h}_r\end{pmatrix} \frac{R^*\begin{pmatrix}1,\dots,r\\ h_1,\dots,h_r\end{pmatrix}}{L} \frac{R^*\begin{pmatrix}1,\dots,r\\ h_1,\dots,h_r\end{pmatrix}}{L}} \qquad (A)$$

where L is a positive number equal to the absolute value of a nonzero r^{th}-order minor of R^*. The denominator of (A) was taken from Equation (8) in the proof of (3.3-1).

Let $\mathcal{B}_{M_R}$ denote the class of bases of M_R. Set

$$\mathcal{B}(E_b) = \{b | b \in \mathcal{B}_{M_R} \text{ and } b \subseteq E_b\}.$$

For any $S = \{e_{i_1},\dots,e_{i_t}\} \subseteq E_b$ we define $[D^{-1}S]$ as

$$[D^{-1}S] = \frac{1}{d_{i_1}} \frac{1}{d_{i_2}} \cdots \frac{1}{d_{i_t}} .$$

Accordingly we can write Δ, the denominator of (A), as

$$\Delta = \sum_{b\in\mathcal{B}(E_b)} [D^{-1}b].$$

The numerator of (A) requires some special attention. A typical term in the numerator of (A) is nonzero if and only if the sets $\{e_{m_1},\dots,e_{m_{r-1}},e_{n-p+i}\}$ and $\{e_{m_1},\dots,e_{m_{r-1}},e_{n-p+j}\}$ are both bases of M_R. Accordingly we define $\mathcal{B}_{i,j}$ as

$$\mathcal{B}_{i,j} = \{S \subseteq E_b | S \cup \{e_{n-p+i}\} \in \mathcal{B}_{M_R} \text{ and } S \cup \{e_{n-p+j}\} \in \mathcal{B}_{M_R}\}$$

for all $1 \le i,j \le p$.

Although the members of $\mathcal{B}_{i,j}$ are in 1-1 correspondence with the nonzero terms in the numerator of (A), there is still the matter of the sign of each term when $i \ne j$. Partition the set $\mathcal{B}_{i,j}$ into two sets $\mathcal{B}^+_{i,j}$ and $\mathcal{B}^-_{i,j}$, where for $i \ne j$

$\mathcal{B}^{+}_{i,j} = \{S \in \mathcal{B}_{i,j} | e_{n-p+i}$ and e_{n-p+j} are oppositely oriented in the circuit $J(S \cup \{e_{n-p+i}\}, e_{n-p+j})$ of $M_{\mathcal{R}}\}$

and

$\mathcal{B}^{-}_{i,j} = \{S \in \mathcal{B}_{i,j} | e_{n-p+i}$ and e_{n-p+j} are similarly oriented in the circuit $J(S \cup \{e_{n-p+i}\}, e_{n-p+j})$ of $M_{\mathcal{R}}\}$

and for $i = j$

$$\mathcal{B}^{+}_{i,i} = \mathcal{B}_{i,i}$$

and

$$\mathcal{B}^{-}_{i,i} = \phi .$$

Theorem (3.4-3): Let $N = (M_{\mathcal{R}}, \mathcal{R}, D; E) \in \mathcal{N}$. *Then*

$$(X_N)_{i,j} = \frac{\sum_{S \in \mathcal{B}^{+}_{i,j}} [D^{-1}S] - \sum_{T \in \mathcal{B}^{-}_{i,j}} [D^{-1}T]}{\sum_{b \in \mathcal{B}(E_b)} [D^{-1}b]} .$$

Proof: To prove (3.4-3) we merely have to show that the sets $\mathcal{B}^{+}_{i,j}$ and $\mathcal{B}^{-}_{i,j}$ correspond to the negative and positive terms, respectively, of the numerator of $(X_N)_{i,j}$ when $i \neq j$.

Let $S \in \mathcal{B}_{i,j}$ and form the base

$$b = S \cup \{e_{n-p+i}\}.$$

Let R^* be a standard representative matrix for $\perp\mathcal{R}$ with respect to b. Consider the submatrix $(j > i)$

$$R^*(b \cup \{e_{n-p+j}\}) = \begin{array}{c} \overbrace{\qquad\qquad\qquad}^{S} \quad e_{n-p+i} \quad e_{n-p+j} \\ \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & x_1 \\ 0 & 1 & \ldots & 0 & 0 & x_2 \\ \vdots & & & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & x_{r-1} \\ 0 & 0 & \ldots & 0 & 1 & x_r \end{bmatrix} \end{array} .$$

The minimal dependent sets of columns of R^* correspond to circuits in $\mathcal{M}_R$ (see Table 2-2). By the definition of $\mathcal{B}_{i,j}$ it follows that $x_r = \pm 1$. Moreover, the set $S \cup \{e_{n-p+i}, e_{n-p+j}\}$ is dependent in $\mathcal{M}_R$ and contains a unique circuit C which has both e_{n-p+i} and e_{n-p+j} as members. Clearly if $x_r = +1$, e_{n-p+i} and e_{n-p+j} are oppositely oriented in C and if $x_r = -1$, e_{n-p+i} and e_{n-p+j} are similarly oriented in C. Moreover, if $x_r = +1$, the corresponding term in $(X_N)_{i,j}$ will be positive and if $x_r = -1$, the corresponding term will be negative. The theorem follows. □

In (3.4-3) we have extended the notion of a "topological" formula to generalized networks, that is, we can evaluate the entries in X_N by a formula which depends only on the resistance-element immittance matrix and the structure of the matroid $\mathcal{M}_R$.

Example 6: Let G be the graph in Figure 16 and $N_Z = (\mathcal{P}(G), I, Z_b; E(G))$, where

$$Z_b = \text{diag}[z_1, z_2, z_3]$$

and

$$\mathcal{B}(E_b) = \{\{e_1, e_2, e_3\}\}.$$

Consequently

$$\Delta = \frac{1}{z_1 z_2 z_3} .$$

To calculate $(X_{N_Z})_{1,1}$ we need $\mathcal{B}_{1,1}$:

$$\mathcal{B}_{1,1} = \{\{e_1, e_3\}, \{e_2, e_3\}\} = \mathcal{B}^+_{1,1}.$$

Therefore

$$(X_{N_Z})_{1,1} = \frac{\dfrac{1}{z_1 z_3} + \dfrac{1}{z_2 z_3}}{\dfrac{1}{z_1 z_2 z_3}} = z_2 + z_1.$$

To calculate $(X_{N_Z})_{1,2}$ we need $\mathcal{B}^+_{1,2}$ and $\mathcal{B}^-_{1,2}$:

$$\mathcal{B}^+_{1,2} = \{\{e_1, e_3\}\}$$

and

$$\mathcal{B}^-_{1,2} = \phi.$$

Therefore

$$(X_{N_Z})_{1,2} = \frac{\dfrac{1}{z_1 z_3}}{\dfrac{1}{z_1 z_2 z_3}} = z_2.$$

Calculating $(X_{N_Z})_{2,2}$ we find

$$(X_{N_Z})_{2,2} = z_2 + z_3.$$

Thus

$$X_{N_Z} = \begin{bmatrix} z_1+z_2 & z_2 \\ z_2 & z_2+z_3 \end{bmatrix}.$$

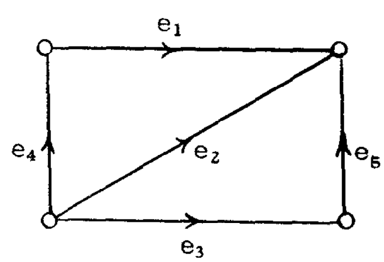

Fig. 16 Graph of Example 6.

We now turn to the special case of generalized networks satisfying $\alpha(E_p) = r(M^*_{\mathcal{R}})$. These networks have special significance in the case of N_Z and N_Y. For instance, if $N = N_Z$, then $\alpha(E_p) = r(M^*_{\mathcal{R}})$ becomes $\alpha(E_p) = r(\mathcal{B}(G))$. Thus the number of port elements coincides with the number of elements in a coforest of G. If, moreover, $N_Z \varepsilon \mathcal{N}$, then E_p contains no bond of G, and consequently E_p is a coforest of G. If $N = N_Y \varepsilon \mathcal{N}$ and $\alpha(E_p) = r(\mathcal{P}(G))$, then E_p is a forest of G.

The above illustrations are encompassed by the following theorem.

Theorem (3.4-4): Let $N = (M_{\mathcal{R}}, \mathcal{R}, D; E) \varepsilon \mathcal{N}$. *Then* $r(M_{\mathcal{R}}^*) = \alpha(E_p)$ *if and only if* E_p *is a base of* $M_{\mathcal{R}}^*$.

Proof: If E_p is a base of $M_{\mathcal{R}}^*$, then $r(M_{\mathcal{R}}^*) = \alpha(E_p)$. Conversely, suppose $r(M_{\mathcal{R}}^*) = \alpha(E_p)$. Since $N \varepsilon \mathcal{N}$, it follows from (3.3-1) that E_p is a base of $M_{\mathcal{R}}^*$. □

Theorem (3.4-5): Let $N = (M_{\mathcal{R}}, \mathcal{R}, D; E) \varepsilon \mathcal{N}$ *and* $r(M_{\mathcal{R}}^*) = \alpha(E_p)$. *Then* $X_N = A\, D\, A^t$, *where A is a totally unimodular matrix.*

Proof: By (3.4-4), E_p is a base of $M_{\mathcal{R}}^*$. Accordingly $E_b = \bar{E}_p$ is a cobase of $M_{\mathcal{R}}^*$. Let R^* be a standard representative matrix of $\mathcal{R}^*$ with respect to the cobase E_b:

$$R^* = [1_{n-p} \,|\, R_p^*].$$

By (2.2-6), R_p^* is a totally unimodular matrix. Calculating X_N using R^* we find that

$$X_N = R_p^{*t}\, D\, R_p^*.$$

Since the transpose of a totally unimodular matrix is totally unimodular, the theorem is proved. □

Under the hypothesis of (3.4-5) we know that X_N is a paramount matrix. In the next theorem we give an additional necessary condition on X_N. Unfortunately these two conditions are not sufficient as we will show in Example 7.

Theorem (3.4-6): Let $Q = [q_{ij}] = ADA^t$, *where* $A = [a_{ij}]$ *is a* $p \times b$ *totally unimodular matrix and D is a* $b \times b$ *diagonal matrix with positive diagonal terms. Then*

$$Q_{i,r,c} = q_{ii} + |q_{rc}| - |q_{ri}| - |q_{ic}| \geq 0,$$

for all $1 \leq i, r, c \leq p$.

Proof: The i,j^{th} element of Q is

$$q_{ij} = \sum_{k=1}^{b} d_k a_{ik} a_{jk}.$$

It is well known [24] that for fixed i and j all the nonzero products $a_{ik}a_{jk}$, for k = 1, ..., b, have the same sign. Consequently

$$|q_{ij}| = \sum_{k=1}^{b} d_k |a_{ik}a_{jk}|.$$

Therefore we can write $Q_{i,r,c}$ as

$$Q_{i,r,c} = \sum_{k=1}^{b} d_k [a_{ik}^2 + |a_{rk}a_{ck}| - |a_{rk}a_{ik}| - |a_{ck}a_{ik}|].$$

We prove $Q_{i,r,c} \geq 0$ by showing that each term in the summation is nonnegative.

Case 1: $a_{ik} = 0$. Therefore the only contribution is from $|a_{rk}a_{ck}|$, which is nonnegative.

Case 2: $a_{ik} \neq 0$ and $a_{rk}a_{ck} = 0$. Therefore the term

$$d_k [a_{ik}^2 - |a_{rk}a_{ik}| - |a_{ck}a_{ik}|]$$

is nonnegative since at least one of the terms $|a_{rk}a_{ik}|$ or $|a_{ck}a_{ik}|$ is zero.

Case 3: $a_{ik} \neq 0$, $a_{rk}a_{ck} \neq 0$. Therefore

$$d_k [a_{ik}^2 + |a_{rk}a_{ck}| - |a_{rk}a_{ik}| - |a_{ck}a_{ik}|] = 0.$$

We conclude therefore that $Q_{i,r,c} \geq 0$ for $1 \leq i, r, c \leq p$. □

Example 7: The matrix

$$Q = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 35 & -30 \\ 0 & -30 & 36 \end{bmatrix}$$

is a *paramount* matrix [24]. Q however does not satisfy (3.4-6) since

$$Q_{2,1,3} = 35 + 0 - 10 - 30 = -5.$$

Accordingly, the paramountcy condition and the condition of (3.4-6) are independent.

Next let Q' be the paramount matrix

$$Q' = \begin{bmatrix} 7 & 1 & 2 & 3 \\ 1 & 12 & 4 & 5 \\ 2 & 9 & 15 & 6 \\ 3 & 5 & 6 & 18 \end{bmatrix}.$$

Cederbaum has shown that Q' cannot be displayed as in the hypothesis of (3.4-6). However $Q'_{i,r,c} \geq 0$ for $1 \leq i, r, c \leq 4$. Consequently, paramountcy and the condition that $Q'_{i,r,c} \geq 0$ are not sufficient to guarantee the unimodular decomposition of Q'.

To conclude this subsection we show that the modified topological matrices introduced by Cederbaum [31] can be extended to generalized networks. In fact, we show that a "modified" matrix exists for a generalized network if $N \varepsilon \mathcal{N}$. There seems to be some ambiguity in the literature as to just when a "modified" matrix exists for a p-port resistance network. For instance, in [31] networks for which modified matrices exist are called "nonsingular" p-port networks and the meaning of nonsingular is left undefined. We show that a modified topological matrix exists if X_N exists.

Let $N = (M_{\mathcal{R}}, \mathcal{R}, D; E) \varepsilon \mathcal{N}$. By (3.3-1) and Table 2-2, line 2, there exists a representative matrix for $\mathcal{R}$ of the following form

$$R = \left[\begin{array}{c|c} R_{11} & 1_p \\ \hline R_{21} & 0_{p' \times p} \end{array}\right] = \mu \times n,$$

where μ = dimension $(\mathcal{R})$. Moreover, by (2.2-6) we can assume that R is a totally unimodular matrix. We define an augmented matrix R^+:

$$R^+ = \left[\begin{array}{c|c|c} R_{11} & 1_p & 0_{p\times p'} \\ \hline R_{21} & 0_{p'\times p} & 1_{p'} \end{array}\right].$$

R^+ is totally unimodular and consequently can be viewed as the representative matrix of a regular vector space $\mathcal{R}^+$. Set

$$N^+ = (M_{\mathcal{R}^+}, \mathcal{R}^+, D; E\cup E')$$

where $E' = \{e_{n+1},\ldots,e_{n+p'}\}$. N^+ is called an *augmented network* for N and $N^+ \in \mathcal{N}$. Set

$$(R^+)^* = [1_{n-p} | -R_{11}{}^t | -R_{21}{}^t].$$

Clearly $(R^+)^*$ is a representative matrix for $\perp\mathcal{R}^+$.

The variables associated with the network N are, as usual,

$$\underline{u} = \begin{bmatrix} \underline{u}_b \\ \hline \underline{u}_p \end{bmatrix}$$

and

$$\underline{w} = \begin{bmatrix} \underline{w}_b \\ \hline \underline{w}_p \end{bmatrix}.$$

The variables associated with N^+ are

$$\underline{u}^+ = \begin{bmatrix} \underline{u}_b \\ \hline \underline{u}_p \\ \hline \underline{u}_{p'} \end{bmatrix} = \begin{bmatrix} \underline{u}_b \\ \hline \underline{u}^+_{p+p'} \end{bmatrix}$$

and

$$\underline{w}^+ = \begin{bmatrix} \underline{w}_b \\ \hline \underline{w}_p \\ \hline \underline{w}_{p'} \end{bmatrix} = \begin{bmatrix} \underline{w}_b \\ \hline \underline{w}^+_{p+p'} \end{bmatrix}.$$

The augmented network N^+ can be viewed as one which is obtained from N by adding p' additional ports to N.

The following two results are easy consequences of the construction of $\mathcal{R}^+$ from $\mathcal{R}$.

Theorem (3.4-7): If $\underline{u}^+ \varepsilon \mathcal{R}^+$, *then* $\underline{u} \varepsilon \mathcal{R}$.

Theorem (3.4-8): If $\underline{w}^+ \varepsilon \perp\mathcal{R}^+$ *and* $\underline{w}_{p'} = \underline{0}$, *then* $\underline{w} \varepsilon \perp\mathcal{R}$.

Theorem (3.4-9) extends the modified topological matrices of p-port resistance networks to generalized networks.

Theorem (3.4-9): Let $N = (M_{\mathcal{R}}, \mathcal{R}, D; E) \varepsilon \mathcal{N}$ *and R be a totally unimodular representative matrix for* $\mathcal{R}$:

$$R = \left[\begin{array}{c|c} R_{11} & 1_p \\ \hline R_{21} & 0_{p'\times p} \end{array}\right] = \mu \times n.$$

Then

$$X_N = \hat{R}\, D\, \hat{R}^t,$$

where

$$\hat{R} = R_{11} - R_{11}\, D\, R_{21}^t [R_{21}\, D\, R_{21}^t]^{-1}\, R_{21}.$$

Moreover, a matrix R in the above form always exists. The matrix $\hat{R}$ *is called a modified topological matrix.*

Proof: From (3.3-2) it follows that

$$X_{N^+} = \left[\begin{array}{c|c} R_{11}\, D\, R_{11}^t & R_{11}\, D\, R_{21}^t \\ \hline R_{21}\, D\, R_{11}^t & R_{21}\, D\, R_{21}^t \end{array}\right] \tag{1}$$

and

$$\underline{w}_p^+ = -\ (X_{N^+})\underline{u}_p^+ . \tag{2}$$

Set

$$\underline{u}^{+}_{p+p'} = \begin{bmatrix} \underline{u}_p \\ \hline \underline{u}_{p'} \end{bmatrix} = \begin{bmatrix} 1_p \\ \hline (R_{21}\, D\, R^t_{21})^{-1}\; R_{21}\, D\, R^t_{11} \end{bmatrix} \underline{u}_p \,. \tag{3}$$

Then $\det[R_{21}\, D\, R^t_{21}] \neq 0$, since the rows of R_{21} are linearly independent.

If we specify $\underline{u}_p$ arbitrarily in (3) and apply $\underline{u}^{+}_{p+p'}$ to N^{+} we get

$$X_{N^+}\, \underline{u}_{p+p'} = \underline{w}^{+}_{p+p'} = \begin{bmatrix} \underline{w}_p \\ \hline \underline{0} \end{bmatrix}. \tag{4}$$

In view of (3.4-7) and (3.4-8), the variables $\underline{u}$ and $\underline{w}$, under the conditions imposed by (3), in N^{+} satisfy the network equations of N. Since both N and N^{+} are in $\mathcal{N}$ the relation between $\underline{w}_p$ and $\underline{u}_p$ in N^{+}, under the constraint (3), is precisely

$$\underline{w}_p = - X_N \underline{u}_p. \tag{5}$$

Next we obtain an alternative expression for $\underline{w}_p$ in terms of $\underline{u}_p$ operating in N^{+} under (3).

Using the constraint (3) and the fact that $\underline{u}^{+} \in \mathcal{R}^{+}$ and $\underline{w}^{+} \in \perp\mathcal{R}^{+}$, it is not difficult to see that

$$R^t_{11}\, \underline{u}_p + R^t_{21}\, \underline{u}_{p'} = \underline{u}_b \tag{6}$$

$$R_{11}\, \underline{w}_b + \underline{w}_p = \underline{0} \tag{7}$$

and

$$R_{21}\, \underline{w}_b = \underline{0}. \tag{8}$$

Combining (6) and (3) we get

$$\underline{u}_b = \hat{R}^t\, \underline{u}_p. \tag{9}$$

Also from (7) and (8) we get

$$\underline{w}_p = - \hat{R}\, \underline{w}_b . \qquad (10)$$

Using $\underline{w}_b = D\,\underline{u}_b$, (9) and (10), we arrive at

$$\underline{w}_p = - [\hat{R}\, D\, \hat{R}^t]\, \underline{u}_p . \qquad (11)$$

Equation (11) describes the relation between the variables $\underline{w}_p$ and $\underline{u}_p$ in N^+ under constraint (3), and (5) describes the identical relation between $\underline{w}_p$ and $\underline{u}_p$. Consequently $X_N = \hat{R}\, D\, \hat{R}^t$.

As was previously indicated, a matrix R of the desired form exists if $N \in \mathcal{N}$. □

3.5 Singular Immittance Matrices

As is well known in the case of p-port resistance networks, when an immittance matrix is singular the linear dependence of the columns (or rows) contains information on the port structure of the network. In this subsection we show how the linear dependence of the columns of X_N is reflected in the structure of the matroid M_R associated with N. More precisely, we show that the circuits of $M_R \times E_p$ are in 1-1 correspondence with the sets of minimal dependent columns of X_N.

Theorem (3.5-4) deals with a converse problem for paramount matrices. In the previous section we showed that if $N \in \mathcal{N}$, then X_N is a paramount matrix. The converse problem is very difficult, that is, given Q, a p × p paramount matrix, determine a generalized network $N \in \mathcal{N}$ (if one exists) satisfying $X_N = Q$. This converse problem is called the *synthesis problem* for generalized networks. In this subsection we prove an interesting result on singular paramount matrices which has bearing on the synthesis problem. We show that the linear dependence of the columns of a singular paramount matrix cannot be arbitrary and in fact its null space must be regular.

Theorem (3.5-1): If $N = (M_R, R, D; E) \in \mathcal{N}$, then $\underline{u}_p^t\, X_N\, \underline{u}_p = \underline{u}_b^t\, D\, \underline{u}_b$.

Proof: Since $\underline{u} \in R$ and $\underline{w} \in \perp R$, it follows that $\underline{u}_b^t \underline{w}_b + \underline{u}_p^t \underline{w}_p = 0$. Using $\underline{w}_b = D\underline{u}_b$ and $\underline{w}_p = -X_N \underline{u}_p$, the theorem follows. □

The next result relates the minimal dependent columns of X_N to the structure of the matroid M_R.

Theorem (3.5-2): Let $N = (M_R, R, D; E) \in \mathcal{N}$. Then a set of columns of X_N forms a minimal dependent set if and only if the corresponding set of elements in E_p is a circuit of M_R.

Proof: Let $C \subseteq E_p$ be a circuit of M_R. Then there exists an elementary vector $\underline{u}' \in R$ such that $\|\underline{u}'\| = C$. Clearly the pair $\underline{u} = \underline{u}'$ and $\underline{w} = \underline{0}$ satisfy the network equations, and since $N \in \mathcal{N}$ it follows that

$$X_N \underline{u}_p' = \underline{0}\ , \tag{1}$$

where $\underline{u}' = \begin{bmatrix} \underline{0} \\ \hline \underline{u}_p' \end{bmatrix}$.

We claim that the columns of (1) which are linearly dependent form a minimal dependent set.

Assume there exists a nonzero vector $\underline{u}_p''$ such that

$$X_N \underline{u}_p'' = \underline{0}$$

and

$$\|\underline{u}''\| \subset \|\underline{u}'\|, \tag{2}$$

where $\underline{u}'' = \begin{bmatrix} \underline{0} \\ \hline \underline{u}_p'' \end{bmatrix}$. Since $N \in \mathcal{N}$, $\underline{u}_p$ can be specified arbitrarily and therefore by (3.5-1) $\underline{u}'' \in R$. But then (2) contradicts the hypothesis and accordingly the dependent columns in (1) form a minimal dependent set.

To show necessity suppose

$$X_N \underline{u}_p = \underline{0} \tag{3}$$

and that the dependent columns of (3) form a minimal dependent set. Again since $N \in \mathcal{N}$, $\underline{u}_p$ can be specified arbitrarily and therefore by (3.5-1) the vector

$$\underline{u} = \begin{bmatrix} \underline{0} \\ --- \\ \underline{u}_p \end{bmatrix}$$

satisfies $\underline{u} \in R$ and $\|\underline{u}\| \subseteq E_p$.

Assume $\underline{u}$ is not elementary. Then there exists a nonzero vector $\underline{v}^t = [\underline{0}^t \ \underline{v}_p^t]$ satisfying

$$\|\underline{v}\| \subseteq \|\underline{u}\|$$

and

$$X_N \underline{v}_p = \underline{0}. \tag{4}$$

However, (4) contradicts the hypothesis and accordingly $\underline{u}$ is elementary. Therefore there exists a circuit $C \subseteq E_p$ such that $C = \|u\|$. □

Theorem (3.5-2) shows that, in the case of the generalized network, matroid theory allows a geometric interpretation of the singular immittance matrices. For the cases N_Z and N_Y, Theorem (3.5-2) specializes to the following well known result.

Theorem (3.5-3): Let Z(Y) be the o.c. impedance (s.c. admittance) matrix of a resistance network whose network graph is G. Then the minimal dependent columns of Z(Y) are in a 1-1 correspondence with the polygons (bonds) of G which are contained in E_p.

Previously we defined what is meant by a primitive (elementary) representative vector $\underline{x}^t$ with respect to some vector space R. It should be clear that if U is a collection of n-tuples $\underline{x}$, then we can use the term primitive (elementary) vector in U without reference to a vector space R. Moreover if U is closed under addition of n-tuples and multiplication by a member of F, then we call U a vector space of n-tuples on E over the field F. The reference to a set E is necessary if, for some $\underline{x} \in U$, the notation $\|\underline{x}\|$ is to have meaning. U (a vector space of n-tuples) is called regular if F is the field of real numbers and corresponding to each elementary vector $\underline{x} \in U$ there exists a primitive vector $\underline{x}' \in U$ satisfying

$$\|\underline{x}'\| = \|\underline{x}\|.$$

In the next theorem we characterize the null space of any paramount matrix. If Q is a p × p matrix, the *null space* N(Q) of Q is the set of all p-tuples $\underline{x}$ which satisfy $Q\underline{x} = \underline{0}$:

$$N(Q) = \{\underline{x} \mid Q\underline{x} = \underline{0}\}.$$

Theorem (3.5-4): Let Q be a p × p paramount matrix; then N(Q), the null space of Q, is a regular vector space of p-tuples on $E_p(\alpha(E_p) = p)$.

Proof: Without loss of generality, assume the first r columns of Q form a minimal dependent set.

Assume there exists a principal minor $Q\begin{pmatrix} i_1, \ldots, i_{r-1} \\ i_1, \ldots, i_{r-1} \end{pmatrix} = 0$, where $1 \leq i_1 < \ldots < i_{r-1} \leq r$. Since Q is paramount, then any $(r-1)^{th}$-order minor using columns $i_1, \ldots, i_{r-1}$ is zero. Accordingly columns $i_1, \ldots, i_{r-1}$ are linearly dependent; but this contradicts the hypothesis. Therefore every $(r-1)^{th}$-order principal minor formed from the first r columns is nonzero.

Let Q_r be the submatrix formed from the first r rows and columns of Q; by hypothesis $\det[Q_r] = 0$. If we let Δ_{ij} be the cofactor obtained from Q_r by crossing out row i and column j, it follows from Jacobi's theorem [36] that

$$\Delta_{ii}\Delta_{jj} = \Delta_{ij}\Delta_{ji} \; . \tag{1}$$

However, Q is paramount and consequently

$$\Delta_{kk} \geq |\Delta_{kh}| = |\Delta_{hk}|, \tag{2}$$

for all $1 \leq k \leq r$ and $1 \leq h \leq r$. Using (1) and (2) and the fact that $\Delta_{kk} \neq 0$ for $1 \leq k \leq r$, we conclude that all the first cofactors of Q_r are equal in absolute value.

It follows from the above analysis that the coefficients of the linear relation of the first r columns of Q can be chosen to be ±1.

Since the first r columns form a minimal dependent set the vector $\underline{x}$, whose coordinates are the coefficients of this linear relation, is elementary in N(Q). Moreover, we have shown that there exists a primitive vector $\underline{x}'$ such that $\|\underline{x}'\| = \|\underline{x}\|$ and $Q\underline{x}' = \underline{0}$. □

Theorem (3.5-4) enables one to exhibit a paramount matrix in a very revealing form.

Theorem (3.5-5): Let Q be a p × p paramount matrix of rank s satisfying $Q\begin{pmatrix}1,\ldots,s\\1,\ldots,s\end{pmatrix} \neq 0$. *Then Q can be expressed as*

$$Q = B^t Q_s B,$$

where B is an s × p totally unimodular matrix and Q_s *is the submatrix formed from the first s rows and columns of Q.*

Proof: Partition Q as

$$Q = \left[\begin{array}{c|c} Q_s & Q_{12} \\ \hline Q_{12}^t & Q_{22} \end{array}\right], \qquad (1)$$

where $Q_s = s \times s$,

$Q_{12} = s \times (p-s)$

$Q_{22} = (p-s) \times (p-s)$.

Set

$$T = \left[\begin{array}{c|c} Q_s^{-1} & 0_{s\times(p-s)} \\ \hline -Q_{12}^t Q_s^{-1} & 1_{p-s} \end{array}\right]$$

and form TQ:

$$TQ = \left[\begin{array}{c|c} 1_s & Q_s^{-1} Q_{12} \\ \hline 0_{(p-s)\times s} & Q_{22} - Q_{12}^t Q_s^{-1} Q_{12} \end{array}\right].$$

Since det[T] $\neq$ 0, the rank of TQ is s and accordingly

$$Q_{22} - Q_{12}^t Q_s^{-1} Q_{12} = 0_{(p-s)\times(p-s)}. \tag{2}$$

Setting $B = [1_s \mid Q_s^{-1} Q_{12}]$ and using (1) and (2) we can express Q as

$$Q = B^t Q_s B. \tag{3}$$

Let $\underline{x}$ be a p-tuple satisfying $Q\underline{x} = \underline{0}$. Then

$$(B^t Q_s)(B\underline{x}) = \underline{0}. \tag{4}$$

The matrix $B^t Q_s$ is $p \times s$ and of rank s and the matrix $B\underline{x}$ is $s \times 1$. Accordingly (4) implies

$$B\underline{x} = \underline{0}.$$

Conversely, if $\underline{x}$ is a p-tuple satisfying $B\underline{x} = \underline{0}$, then $Q\underline{x} = \underline{0}$.

The above analysis shows that

$$N(Q) = \{\underline{x} \mid B\underline{x} = \underline{0}\}. \tag{5}$$

It should be clear from the construction of B and Equation (5) that the row space of B^* is precisely the transpose of the vectors in N(Q), where

$$B^* = [-Q_{12}^t Q_s^{-1} \mid 1_{p-s}].$$

(Note that $B^* B^t = 0_{(p-s)\times s}$.) By (3.5-4) N(Q) is regular and therefore the theorem follows using (2.4-4) and (2.2-6). □

An open question for generalized networks is to determine whether for every paramount matrix Q there exists a generalized network N such that $X_N = Q$. The case of the singular paramount matrix poses an interesting test in view of (3.5-2). Consequently, if we conjecture that N exists for any Q, then Theorem (3.5-4) must also be true. Since we have in fact shown (3.5-4) to be true, the question remains an open one. This problem involves the principle of duality and is therefore considered in more detail in the following subsection.

3.6 *Duality*

Duality in electrical networks has to do with the relationship between the admittance and impedance formulations of electrical networks. One might erroneously conclude that, with respect to p-port resistance networks, theorems about impedances are immediately true for admittances and vice versa. Cederbaum [23] has shown the existence of a paramount matrix which is the o.c. impedance matrix of some 4-port resistance network but is *not* the s.c. admittance matrix for any 4-port resistance network. In this subsection we will discuss the notion of duality in generalized and p-port resistance networks.

Let $N = (M_{\mathcal{R}}, \mathcal{R}, D; E)$. We define N^*, the *dual network* of N as

$$N^* = (M_{\perp\mathcal{R}}, \perp\mathcal{R}, D^{-1}; E).$$

Theorem (3.6-1): Let $N, N^* \varepsilon \mathcal{N}$. *Then* $(X_N)^{-1} = X_{N^*}$.

Proof: Theorem (3.6-1) is easily seen to be true by comparison of the network equations for N and N^*. □

Every network (in $\mathcal{N}$ or not in $\mathcal{N}$) has a dual network.

A paramount matrix Q is said to be *realized by* N if $Q = X_N$. The network N is called a *realization* of Q.

Two networks $N^{(1)}$ and $N^{(2)}$ are said to be equivalent if $X_{N(1)} = X_{N(2)}$.

We denote by [N], the class of all networks equivalent to N, that is, the set of all networks which realize Q is equal to [N], where $X_N = Q$. The equivalence class [N] is sometimes denoted by $E(Q)$, where Q is the paramount matrix satisfying $Q = X_N$. We point out that in this discussion the set E is not fixed and therefore if

$$N^{(1)} = (M_{\mathcal{R}(1)}, \mathcal{R}^{(1)}, D^{(1)}; E^{(1)})$$

and

$$N^{(2)} = (M_{\mathcal{R}(2)}, \mathcal{R}^{(2)}, D^{(2)}; E^{(2)})$$

are two different networks in [N], it is not necessary that $E^{(1)} = E^{(2)}$. However, $\alpha(E_p^{(1)}) = \alpha(E_p^{(2)})$.

A conjecture that so far the authors have been unable to prove or disprove is

Theorem (3.6-2): (Conjecture) Let Q be a p × p paramount matrix; then $E(Q) \neq \phi$.

This conjecture, in its relation to singular paramount matrices, was mentioned at the end of the preceding subsection.

Notice that in Figure 10 the class of regular matroids partitions into four sets:

(i) H : nonplanar matroids - Type H.
(ii) K : nonplanar matroids - Type K.
(iii) P : planar matroids.
(iv) O : regular matroids not in H, K or P.

Let Q be a p × p paramount matrix and suppose $E(Q) \neq \phi$. Then for each $N \in E(Q)$, the matroid M_R associated with N is in exactly one of the classes H, K, P or O. Therefore we can partition $E(Q)$ into four sets: H(Q), K(Q), P(Q) and O(Q), where

$H(Q) = \{N \mid N \in E(Q)$ and the matroid associated with N is in H.$\}$
$K(Q) = \{N \mid N \in E(Q)$ and the matroid associated with N is in K.$\}$
$P(Q) = \{N \mid N \in E(Q)$ and the matroid associated with N is in P.$\}$
$O(Q) = \{N \mid N \in E(Q)$ and the matroid associated with N is in O.$\}$

We have the following very important results based on Subsection 2.6.

Theorem (3.6-3): Let Q be a fixed paramount matrix and suppose $E(Q) \neq \phi$ *and* $E(Q)$ *is partitioned into the sets H(Q), K(Q), P(Q) and O(Q) defined above. Then Q is the o.c. impedance (s.c. admittance) matrix of some network if and only if at least one of the sets K(Q) or P(Q) (H(Q) or P(Q)) is nonempty.*

Theorem (3.6-4): Let Q be a fixed paramount matrix and suppose $E(Q) \neq \phi$ *and that* $E(Q)$ *is partitioned into the sets H(Q), K(Q), P(Q) and O(Q). Then Q is both the o.c. impedance matrix of some network and the s.c. admittance matrix of some network if and only if at least one of the sets H(Q) or P(Q) is nonempty and at least one of the sets K(Q) and P(Q) is nonempty.*

From (3.6-4) we obtain the well known results that if Q has a planar network realization $N_Z = (P(G), I, Z_b; E(G))$, then there is a graph G' such that $N_Y = (\mathcal{B}(G'), V', Y_b'; E(G'))$ and

$$X_{N_Y} = X_{N_Z}.$$

Obviously G' can simply be taken to be G^* the dual graph of G and $Y_b' = Z_b$. We state this as a theorem.

Theorem (3.6-5): Let Q be a fixed paramount matrix and suppose $E(Q) \neq \phi$ *and* $E(Q)$ *is partitioned into the sets* $H(Q)$, $K(Q)$, $P(Q)$ *and* $O(Q)$. *Then if* $P(Q) \neq \phi$, *there exists at least two networks* N_Z *and* N_Y *in* $E(Q)$ *such that* $X_{N_Z} = X_{N_Y}$.

The above results are well known but to the best of the authors' knowledge have not been previously presented in the context of the theory of regular matroids.

In the remainder of this subsection we discuss the notion of duality in electrical networks. As we stated in Subsection 2.5, duality, in general, implies that we have two sets of (dual) quantities and operations such that if a theorem is proved in terms of one set, then the same theorem with dual quantities inserted everywhere yields a true theorem. We have indicated how matroid theory forms a rigorous basis for a duality theory for graphs and in Table 2-3 we have listed the dual concepts for graphs and the corresponding matroid-theoretic quantities.

Since our generalized networks are based on matroid-theoretic ideas, the specialization of theorems on generalized networks to theorems on p-port resistance networks induces a duality theory for p-port resistance networks. Theorems (3.3-4) and (3.5-3) are examples of how specialization of matroid-theoretic results leads to two graph-theoretic results in each case. Notice also that in (3.3-4) it is essential to replace the dual operations as well as the dual quantities.

We can say, in the context of our definition of duality (see Subsection 2.5), that Z and Y are dual quantities. The extent of this claim is the following. If one proves a theorem for X_N, the immittance matrix of a generalized network in $\mathcal{N}$, then the theorem is immediately true in terms of any X_{N_Z} and X_{N_Y}, where $N_Z, N_Y \in \mathcal{N}$. Thus we have a precise meaning for duality and can accordingly determine whether an appeal to duality is justified in any particular case.

Many books on network theory present "duality" concepts from a much narrower point of view than discussed here and have consequently fostered some incorrect notions. We present a typical approach to "duality." Let $N_Y = (\mathcal{B}(G), V, Y_b; E(G))$ and $N_Z = (\mathcal{P}(G^*), I, Z_b; E(G^*))$ where G^* is a dual graph of G (G is necessarily a planar graph). The network equations for N_Y and N_Z are shown in Table 3-3. In Table 3-3 we use the notation V(G) and I(G) for the coboundary and 1-cycle spaces, respectively, of G. Since G^* is a dual graph of G, $I(G^*) = V(G)$ and $V(G^*) = I(G)$ and consequently, if $Z_b = Y_b$, then $X_{N_Y} = X_{N_Z}$. Clearly, if we restrict our attention to networks whose network graphs are planar, then we can make the statement that impedance and admittance are *indistinguishable* quantities, not dual quantities. The duality of electrical networks as depicted in Table 2-3 is not one in which impedance and admittance blend into one concept; on the contrary, impedances and admittances are different but dual quantities in general.

N_Y (Network Equations)	N_Z (Network Equations)
$\underline{i}_b = Y_b \underline{v}_b$	$\underline{v}_b = Z_b \underline{i}_b$
$\underline{v} \in V(G)$	$\underline{i} \in I(G^*)$
$\underline{i} \in I(G)$	$\underline{v} \in V(G^*)$

Table 3-3 Planar Networks.

As we pointed our previously, matroid theory forms the rigorous basis of the duality within the same graph. This concept of duality is so important that we feel some further discussion is justified since all too often appeals to duality are made and no justification is given. We consider the following problem in order to make a point. Suppose we were given Table 2-3 and told that columns 2 and 3 were "dual" quantities but we were unaware of the corresponding matroid-theoretic concepts. Then if we were asked to determine under what conditions we would be justified in appealing to duality for a rigorous proof of a dual theorem, what must we do?

There is one approach to this problem which is by far the most widely used and does not actually use duality to prove the dual theorem. Essentially, the technique is to consciously, or unconsciously, substitute the dual quantities into the proof of the original theorem and make the observation that the proof goes through in the dual case; that is, one *proves* the dual theorem without an appeal to duality. In effect one must prove *two* theorems, the original and the "dual" theorem.

A more sophisticated approach to this question would be to prove a theorem concerning the dual quantities, a theorem which justifies an appeal to duality. Such a theorem would necessarily set down the precise conditions under which one could use duality as a rigorous proof of a "dual" theorem. To establish such a rigorous basis for Table 2-3 we would, as we mentioned at the beginning of this paper, have to invent matroid theory since it *is* the formal basis of the dual nature of the same graph. Matroids are a generalization of *both* the bond and polygon concepts and the axioms of matroids are the "rules" under which proofs of theorems must be carried out if duality is justified. Thus we state again that formal duality relies on a single concept within which theorems can be proven and subsequently specialized to two (or more) specific cases.

Accordingly, we should be able to detect whether an appeal to duality in any particular case is justified. For example, suppose it is asserted that, since Z and Y are dual quantities, if a paramount matrix Q can be realized by N_Z, then there exists a network N_Y which also realizes Q. This kind of claim has no basis in our definition of duality. Clearly, there is no appeal to a matroid-theoretic theorem which can be specialized in *two* ways. There isn't even an appeal to a theorem proven in terms of the polygon (bond) concepts which can be given a dual interpretation. Accordingly, such an assertion connot be based on duality concepts. Theorem (3.6-3) shows when a paramount matrix can be the o.c. impedance (s.c. admittance) matrix of some network. The conditions are, of course, dual conditions.

Theorem (3.6-5) gives a sufficient condition for two networks N_Y and N_Z to have identical immittance matrices. However, consider the following: suppose $P(Q) = \phi$ and $H(Q) \neq \phi$, that is, Q is a $p \times p$ paramount matrix which is only realized by networks whose network graphs are nonplanar. An important problem is to determine whether (i) $K(Q) = \phi$ in general, (ii) $K(Q) \neq \phi$ in general, or (iii) that, depending on Q, either (i) or (ii) is possible. Duality cannot be expected to answer this question or similar ones. A theory of equivalent networks must be developed in order to understand the issues involved. Such a theory is presently nonexistent.

4. CONCLUSION

We have defined a new concept, namely, a network based on a matroid which we call a generalized network, and then have applied this concept to the problems of network analysis and synthesis. Unlike networks based on graphs, an important property of generalized networks is that they satisfy the principle of duality. To make the paper self-contained we have given in Part I an introduction to basic matroid theory. Using this theory we have then set up the analysis problem for generalized networks and finally considered the synthesis of p-port resistance networks. We derive new results in both network analysis and synthesis and generalize some old results for networks on graphs. It is believed that the formulation of the concept of the generalized network and the setting up of the network analysis problem for these networks will yield computationally efficient analysis techniques for RLC networks. Finally, the presentation of the synthesis problem for the p-port resistance network in terms of generalized networks may aid in solving this crucial problem.

REFERENCES

1. Whitney, H., "On the Abstract Properties of Linear Dependence," *Amer. J. Math.*, 57, 1935, pp. 509-533.
2. Tutte, W. T., "Matroids and Graphs," *Trans. Amer. Math Soc.*, Vol. 90, March 1959, pp. 527-552.
3. Tutte W. T., "A Homotopy Theorem for Matroids, I," *Trans. Amer. Math Soc.*, Vol. 88, 1958, pp. 144-160.
4. Tutte, W. T., "A Homotopy Theorem for Matroids, II," *Trans. Amer. Math. Soc.*, Vol. 88, 1958, pp. 161-174.
5. Welsh, D., "On the Hyperplanes of a Matroid," *Proc. of Camb. Phil. Soc.*, Vol. 65, No. 11, 1969, pp. 11-18.
6. Crapo, H. H. and G. C. Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, MIT Press, Cambridge, Massachusetts, 1970.
7. Edmonds, J., "Minimum Partition of a Matroid into Independent Subsets," *J. Res. Nat. Bur. Std.*, Vol. 69B, 1965, pp. 67-72.
8. Edmonds, J., "Lehman's Switching Game and a Theorem of Tutte and Nash-Williams," *J. Res. Nat. Bur. Std.*, Vol. 69B, 1965, pp. 73-77.

9. Edmonds, J. and D. Fulkerson, "Transversals and Matroid Partition," *J. Res. Nat. Bur. Std.*, Vol. 69B, No. 3, July-September 1965.

10. Lehman, A., "A Solution to the Shannon Switching Game," *J. Soc. Indust. Appl. Math.*, Vol. 12, No. 4, December 1964.

11. Tutte, W. T., "Menger's Theorem for Matroids," *J. Res. Nat. Bur. Std.*, Vol. 69B, 1965, pp. 49-53.

12. Weinberg, L., "Planar Graphs and Matroids," in *Graph Theory and Applications*, edited by Y. Alavi, D. R. Lick, and A. T. White, Springer-Verlag, New York, 1972, pp. 313-329.

13. Edmonds, J., "Submodular Functions, Matroids and Certain Polyhedra," in *Combinatorial Structures and Their Applications, (Proc. Calgary International Conference,* 1969), Gordon and Breach, New York, 1970, pp. 69-87.

14. Young, P. Y., "Equicardinal Matroids and Matroid Designs," Ph.D. Thesis, University of Michigan, 1970.

15. Minty, G., "On the Axiomatic Foundations of the Theories of Directed Linear Graphs, Electrical Networks, and Network Programming," *J. Math. Mech.*, Vol. 15, No. 3, March 1966, pp. 485-520.

16. Bruno, J., "Matroids, Graphs and Resistance Networks," Ph.D. Thesis, City College of New York, 1969.

17. Trent, H., "On the Connection Between the Properties of Oriented Linear Graphs and the Analysis of Lumped Physical Systems," *J. Res. Nat. Bur. Std.*, Vol. 69B, 1965, pp. 79-84.

18. Kron, G., "Graphs as Illegitimate Models of Electrical Networks," *Matrix and Tensor Quarterly*, September 1961, pp. 1-10.

19 Weinberg, L., *Network Analysis and Synthesis*, R. E. Krieger Publishing Co., Inc., Huntington, New York, 1975.

20. Cederbaum, I., "Paramount Matrices and Synthesis of Resistive N-Ports," *IRE Trans. on Circuit Theory*, March 1961, pp. 28-31.

21. Boesch, F. T., "Some Aspects of the Analysis and Synthesis of Resistor Networks," in *Networks and Switching Theory - The NATO Advanced Study Institute*, edited by G. Biorci, Academic Press, New York, 1968.

22. Iri, M., "Multitrees and Necessary Conditions for the Realization of a Conductance Matrix," *RAAG Research Notes*, Third Series, No. 89, June 1965.

23. Cederbaum, I., "Topological Considerations in the Realization of Resistive n-port Networks," *IRE Trans. on Circuit Theory*, Vol. CT-8, September 1961, pp. 324-329.

24. Cederbaum, I., "Applications of Matrix Algebra to Network Theory," *IRE Trans. on Circuit Theory*, Vol. CT-6, (Special Supplement), May 1959, pp. 152-157.

25. Guillemin, E. A., "On the Realization of an n^{th}-Order G Matrix," *IRE Trans. on Circuit Theory*, Vol. CT-8, September 1961, pp. 318-323.

26. Boesch, F. T. and D. C. Youla, "Synthesis of n+1 Node Resistor n-ports," *IEEE Trans. on Circuit Theory*, Vol. CT-12, No. 4, December 1965, p. 515.

27. Biorci, G. and P. Civalleri, "On the Synthesis of Resistive n-port Networks," *IRE Trans. on Circuit Theory*, Vol. CT-8, March 1961, p. 22.

28. Nambiar, K. K., "Answer to Foster's Open Question," *IEEE Trans. on Circuit Theory*, Vol. CT-10, March 1963, pp. 126-127.

29. "The Realization of n-port Networks Without Transformers - A Panel Discussion," *IRE Trans. on Circuit Theory*, Vol. CT-9, September 1962, p. 202.

30. Seshu, S. and M. Reed, *Linear Graphs and Electrical Networks*, Addison-Wesley, Reading, Massachusetts, 1961.

31. Cederbaum, I., "On Equivalence of Resistive n-port Networks," *IEEE Trans. on Circuit Theory*, Vol. CT-12, September 1965, pp. 338-344.

32. Nambiar, K. K., "On the Realization of Singular R Matrices," *IEEE Trans. on Circuit Theory*, September 1964, pp. 421-423.

33. Tutte, W. T., *Introduction to the Theory of Matroids*, Amer. Elsevier Publ. Co., New York, 1971.

34. Tutte, W. T., "Lectures on Matroids," *J. Res. Nat. Bur. Std.*, Vol. 69B, Nos. 1 & 2, January-June 1965.

35. Gantmacher, F. R., *Matrix Theory*, Two Volumes, Chelsea Publ. Co., New York, 1959.

36. Turnbull, H. W., *The Theory of Determinants, Matrices and Invariants*, Dover Publications, Inc., 1960.

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